Propagation Path Length Variations Due to Bending of Optical Fibers

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The propagation path length variation due to bending of optical fibers is analyzed in this article. Both the geometric effect and material stress effects are included in the analysis. These calculations put an upper limit on the expected phase shift in single mode fibers. The fractional change in propagation constant is given by

$$\frac{\delta\beta}{\beta} = (0.15)\left(\frac{a}{\overline{R}}\right)^2,$$

where a is the core radius of the fiber (either single or multimode), and $\overline{R}$ is the bending radius of curvature. Moding effects in multimode fibers cause extraneous phase shifts of unusually high magnitude. This does not occur in single mode fibers, rendering them very insensitive to bending with a theoretical limit given by the above relation.

I. Introduction

To transmit a time and frequency standard with accuracies required for such applications as VLBI (very long baseline interferometry), both a stable transmission path *and* a sophisticated electronic compensation system are required. The choice of a relatively stable transmission medium can greatly alleviate the stringent requirements for the electronic compensation system. Contenders for the transmission path include coaxial system, microwave, and optic-fiber system. Preliminary phase noise measurements on a 3-km multimode fiber-optics link indicate that its performance surpasses all of the other available systems (Ref. 1). It was also observed that bending the optical fiber introduces noticeable phase shift in the transmitted RF signal. The purpose of this article is to analyze this phenomenon and to establish a limit on the amount of RF phase shift introduced due to bending of single-mode fibers. This same calculation procedure cannot be applied to multimode fibers. It has been observed experimentally that bending a multimode fiber introduces unusually high phase shift in the carried RF signal which cannot be accounted for by changes in propagation constants alone. The reason, we believe, is due to moding effects. Multimode fibers carry a large number (several hundred) of transverse modes, some tightly held, some loosely held, and some are even cladding modes. These different modes have different propagation constants (which, in fact, is the origin of dispersion in multimode fibers). Bending the fiber causes a redistribution of the power contained in each mode, with some tightly held modes converted into loosely held or cladding modes. If these loose modes were allowed to enter the receiver, a very large phase shift will be observed, because the effective propagation constant between the receiver and the point when the bend occurs is changed. However, if the loosely held or the cladding modes were lost before they enter the receiver, no effect should be seen on the RF phase shift. This proposition is supported by the experimental observation that, while bending the fiber within several hundred meters of the receiver produces several degrees of phase changes in a 100-MHz RF signal, bending the fiber more than 1 km away from the receiver does not produce any significant effect on the phase.

The above phenomenon does not occur in single-mode fibers because they carry only one mode. Experimentally, bending a single-mode fiber anywhere along the link does not produce any noticeable phase shifts on the signal (as observed with a 100-MHz RF signal on a vector voltmeter with a phase resolution of 0-1 degree). The analysis below gives an upper limit on the amount of phase shift expected in single-mode fibers.

II. Field Solutions of a Straight Optical Fiber

Figure 1 shows a step-index fiber cross section and the refractive index variation. Typical core dimension for a multimode fiber is 50 μm diameter; for a single-mode fiber it is about 5 to 10 μm diameter. The refractive index difference ($n_{core} - n_{clad}$) between the core and cladding material is typically of the order of 10^{-3}. A multimode fiber supports hundreds of transverse modes; in a single-mode fiber all but one of these modes are beyond cutoff. The fundamental mode (HE_{11} mode) that propagates in a single-mode fiber theoretically does *not* have a cutoff frequency (in contrast to a hollow metallic waveguide), but for sufficiently low optical frequency the power contained in the core is so small that for all practical purposes it can be regarded as beyond cutoff.

Electromagnetic wave propagation inside a dielectric fiber is governed by the wave equation:

$$\nabla^2 \mathbf{E} + k^2 n^2(x,y,z)\, \mathbf{E} = 0 \tag{1}$$

where **E** is the electric field vector, n is the refractive index and k is the free space propagation constant $= 2\pi/\lambda$. For weakly guiding fibers (small difference between cladding and core index) the fields are very nearly uniformly and linearly polarized (Ref. 2) so that a scalar wave equation, obtained by replacing the **E** vector by a scalar quantity ψ, suffices to describe the modal behavior. As illustrated in Fig. 1, propagation is in the y direction, giving rise to a factor $e^{-i\beta y}$ in the field ψ, and the transverse mode pattern can be solved from Eq. (1) subjected to pertinent boundary conditions. They can be represented in cylindrical coordinates by the Bessel functions:

$$\psi_l(r,\phi) = \begin{Bmatrix} J_l(ur/a) \;/\; J_l(u) \\ \\ K_l(ur/a)\;/K_l(u) \end{Bmatrix} \cos l\phi \;\cdot \qquad \begin{matrix} r < a \\ \\ r > a \end{matrix} \tag{2}$$

where r and ϕ are the radial and azimuthed coordinates on a cross section of the fiber, J_l and K_l are respectively the Bessel and modified Bessel functions of order l, l is a positive integer describing the mode order, and u and w are a pair of parameters related to the propagation constant β through a set of transcendental characteristic equations. These field solutions have been extensively computed and well documented (Refs. 3, 4).

The problem of solving the field inside a bent fiber is considerably more complicated. The circular symmetry, which enables closed form solutions to be written down as in the case of a straight fiber, no longer exists when the fiber is bent. Previous analyses on bending effects concentrate on radiation loss (Refs. 5, 6). It was shown that such bending loss is negligible if the bend radius is larger than a few centimeters (Ref. 3). This would be assumed in the following analysis.

III. Analysis of a Bent Optical Fiber by Conformal Transformation

Figure 2 shows a sketch of the top view of a bent fiber. With the help of conformal transformation (Ref. 7), a bent section of fiber can be transformed into a straight section with a modified index of refraction profile. For large radius of bend curvature, this modified index profile differs only slightly from the unmodified (straight fiber) one, so that a perturbation technique can be used to evaluate the change in propagation constant due to bending.

The bent fiber, as illustrated in Fig. 2a, lies on the x-y plane. Define a complex number $Z = x + iy$, and a complex function

$$w = u + iv = \overline{R}\, ln\left(\frac{Z}{\overline{R}}\right) \tag{3}$$

which maps every point on the x-y plane onto a point in the u-v plane. Under this transformation, a circular annulus as shown in Fig. 2a will be transformed into a straight section as shown in Fig. 2b. The equation which describes wave propagation in the u-v plane is obtained by applying a similar coordinate transformation, Eq. (2), on the wave Eq. (1), and results in

$$\frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2} + \left(\frac{\partial^2 \psi}{\partial z^2} + k^2 n^2 \psi\right) \left|\frac{dZ}{dw}\right|^2 = 0 \tag{4}$$

On the original x-y plane, the wave propagates along the circular arc of the bent fiber governed by the wave equation (1). On the transformed u-v plane, the wave propagates along the v direction of the transformed straight fiber, governed by the transformed wave equation (4). The factor $|dZ/dw|^2$ can be easily evaluated from Eq. (2):

$$\left|\frac{dZ}{dw}\right| = \exp(u/\bar{R}) \tag{5}$$

Not unexpectedly, as the bend radius $\bar{R}$ goes to ∞, $|dZ/dw|$ goes to 1 and Eq. (4) is the same as that for a straight fiber, resulting in mode solutions given in Eq. (2).

IV. Modification of Index Profile Due to Bending Stress

The conformal transformation technique illustrated above takes care of the geometric factor due to bending. However, there is a material factor due to the stress and strain induced in the bent fiber. Obviously, when a fiber is bent, the inner part is compressed and the outer part rarefied. We can assume that at the mean radius of bending $\bar{R}$ (along the axis of the bent fiber) the density of the fiber material is unchanged, and that the local density of the fiber material is inversely proportional to R, the local radius of bending.

To calculate the variation of the refractive index due to a variation in fiber material density, we use the Clausius-Mosotti relation (Ref. 8) for the refractive index of dense material:

$$n^2 = 1 + \frac{N\alpha}{1 - \left(\frac{N\alpha}{3}\right)} \tag{6}$$

where n = refractive index, N is the number of atoms/unit volume of the medium, and α is the atomic polarizability. N is inversely proportional to the local radius of curvature R:

$$N = \frac{\bar{R}}{R} N_0 \tag{7}$$

where N_0 is the material density of the fiber without bending, which corresponds to a refractive index n_0 of about 1.5:

$$n_0^2 = 1 + \frac{N_0\alpha}{1 - \frac{N_0\alpha}{3}} \simeq (1.5)^2 \tag{8}$$

The refractive index at a point in the fiber where the radius of bending is $R = \bar{R} + \rho$ is given by substituting Eq. (7) into (6), expanding in a Taylor series, and assuming $\rho \ll \bar{R}$:

$$\begin{aligned}
n^2 &= 1 + \frac{\left(1 - \frac{\rho}{\bar{R}} + \left(\frac{\rho}{\bar{R}}\right)^2 + \cdots\right)\alpha N_0}{1 - \frac{N_0\alpha}{3}\left(1 - \frac{\rho}{\bar{R}} + \left(\frac{\rho}{\bar{R}}\right)^2 + \cdots\right)} \\
&= 1 + \left(\frac{\alpha N_0}{1 - \frac{N_0\alpha}{3}}\right)\left(1 - \frac{\rho}{\bar{R}} + \left(\frac{\rho}{\bar{R}}\right)^2 + \cdots\right) \\
&\quad \cdot \left(1 - \frac{N_0\alpha}{3}\left(\frac{\rho}{\bar{R}} - \left(\frac{\rho}{\bar{R}}\right)^2 + \cdots\right) + \frac{\left(\frac{N_0\alpha}{3}\right)^2\left(\frac{\rho}{\bar{R}}\right)^2}{\left(1 - \frac{N_0\alpha}{3}\right)^2} + \cdots\right) \\
&= n_0^2\left(1 - \frac{\rho}{\bar{R}}\left(1 + \frac{\frac{N_0\alpha}{3}}{1 - \frac{N_0\alpha}{3}}\right) + \left(\frac{\rho}{\bar{R}}\right)^2\left(1 + \frac{\frac{N_0\alpha}{3}}{1 - \frac{N_0\alpha}{3}}\right)^2 + \cdots\right)
\end{aligned} \tag{9}$$

Using the actual numerical values,

$$n^2(\rho) = n_0^2\left(1 - \frac{\rho}{\bar{R}}(1.4167) + \left(\frac{\rho}{\bar{R}}\right)^2(2.0069) + \cdots\right) \tag{10}$$

where it is understood that $n_0 = n_{core}$ inside the core and $n_0 = n_{clad}$ inside the cladding.

Since the conformal transformation Eq. (3) transforms the radius ρ into the u-coordinate, Eq. (10) can be substituted directly into the wave equation (4) in the transformed u-v plane:

$$\frac{\partial^2\psi}{\partial u^2} + \frac{\partial^2\psi}{\partial v^2} + \left(\frac{\partial^2\psi}{\partial z^2} + k^2 n^2(u)\psi\right)\left|\frac{dZ}{dw}\right|^2 = 0 \tag{11}$$

V. Perturbation Calculation of the Propagation Constant

As mentioned at the end of Section III, when the bending radius $\bar{R}$ is large, $|dZ/dw|$ will be very close to 1, and we can assume that the transverse mode profiles themselves are not significantly modified. We can, however, calculate the change in propagation constant of each mode due to bending. Perturbation theory gives the following first order corrections $\delta\beta_l$ to the propagation constant β_l of the lth mode (Appendix):

$$\delta\left(\beta_l^2\right) = k^2 \int_A |\psi_l|^2 \left(\left|\frac{dZ}{dw}\right|^2 n^2(u) - n_0^2\right) dA \tag{12}$$

where ψ_l is the lth transverse mode profile as given in Eq. (2), $|dZ/dw|$ is given in Eq. (5), and $n^2(u)$ in Eq. (10). The area integral of Eq. (12) is evaluated over the cross section A of the fiber, namely the uz plane. Substitution for the various quantities in Eq. (12) yields

$$\begin{aligned}\delta\left(\beta_l^2\right) &= k^2 \iint_A |\psi_l(u,z)|^2\, n_0^2 \left[e^{2u/\bar{R}}\left(1 - 1.4167\frac{u}{\bar{R}} + 2.0069\left(\frac{u}{\bar{R}}\right)^2 + \cdots\right)^{-1}\right] du dz \\ &\simeq k^2 n_0^2 \iint_A |\psi_l(u,z)|^2 \left(\frac{u}{\bar{R}}(0.5833) + \left(\frac{u}{\bar{R}}\right)^2 (1.1736) + \cdots\right) du dz\end{aligned} \tag{13}$$

In the following, we shall compute the change in propagation constant in single-mode fibers, using the above Eq. (13).

Only one mode propagates in a single-mode fiber: the HE_{11} mode. This mode is circularly symmetric, and is given by Eq. (2) with $l = 0$. Moreover, it can be very closely approximated by a single Gaussian function (Ref. 9) to within 1% error:

$$\psi_0(r,\phi) = \frac{1}{b}\sqrt{\frac{2}{\pi}}\, e^{-r^2/b^2} \tag{14}$$

where the mode width b is given by

$$\frac{b}{a} = 0.65 + \frac{1.619}{V^{3/2}} + \frac{2.879}{V^6} \tag{15}$$

a is the radius of the fiber core, V is the normalized frequency

$$V = \left(n_{core}^2 - n_{clad}^2\right)^{1/2} a^2 \left(\frac{2\pi}{\lambda}\right)^2 \tag{16}$$

and λ is the free space wavelength. Substituting Eq. (14) into Eq. (13) gives

$$\delta\left(\beta_0^2\right) = k^2 n_0^2 \iint_A \frac{1}{w^2}\frac{2}{\pi} e^{-2r^2/b^2}\left(\frac{u}{\bar{R}}(0.5833) + \left(\frac{u}{\bar{R}}\right)^2 (1.1736) + \cdots\right) dA \tag{17}$$

$$= k^2 n_0^2 \int_0^{2\pi}\int_0^{\infty} \frac{1}{b^2}\frac{2}{\pi} e^{-2r^2/b^2}\left(\frac{r\cos\phi}{\bar{R}}(0.5833) + \frac{r^2\cos^2\phi}{\bar{R}^2}(1.1736) + \cdots\right) r dr d\phi \tag{18}$$

$$= 1.1736\frac{k^2 n_0^2}{b^2}\int_0^{\infty} e^{-2r^2/b^2}\frac{r^3}{\bar{R}^2} dr + \text{higher order terms in } \frac{1}{\bar{R}} \tag{19}$$

$$\delta(\beta^2) = 0.293\, k^2 n_0^2 \left(\frac{b}{\bar{R}}\right)^2 \tag{20}$$

$$\frac{\delta\beta}{\beta} = 0.147\left(\frac{b}{\bar{R}}\right)^2 \tag{21}$$

The fractional change in the propagation constant is thus proportional to the square of the ratio of mode width b to bending radius R. According to Eq. (15), the width b of the fundamental mode is approximately equal to the core radius, which is about 5 μm for single-mode fiber. Eq. (21) can thus be rewritten as

$$\frac{\delta\beta}{\beta} \simeq 4 \times 10^{-8}\left(\frac{1}{\bar{R}\text{ (in cm)}}\right)^2 \tag{22}$$

A bending radius of, say, 4 cm will thus cause a change in the propagation constant of 1 part in 10^8 in single-mode fibers.

VI. Conclusion

Measurements on an experimental multimode fiber link indicate that fiber-optics is the most appropriate means for transmitting time and frequency standards. The above calculations indicate that single-mode fibers can be very insensitive to bending perturbations. However, the problem still remains as to how to efficiently couple the laser source to a single-mode fiber in a convenient, compact and noncritical way. With very tight focusing and critical alignments, a coupling coefficient of over 50% has been achieved (Ref. 10). However, typical coupling loss in actual single-mode fiber systems amounts to 10 dB or more (Ref. 11), compared to an easily achieved 3-dB coupling loss in multimode fiber systems. The type of lasers used is also crucial in determining the coupling coefficient. Since the mode in single-mode fibers is symmetric, a laser with a symmetric output field would facilitate coupling. Means of achieving noncritical coupling into a single-mode fiber (such as tapers) are currently under investigation. Also under investigation are means to reduce moding effects in multimode fibers.

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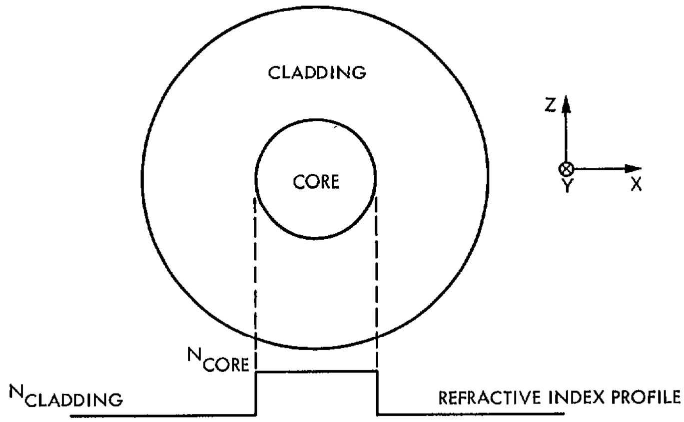

Fig. 1. Cross section of a step index optical fiber

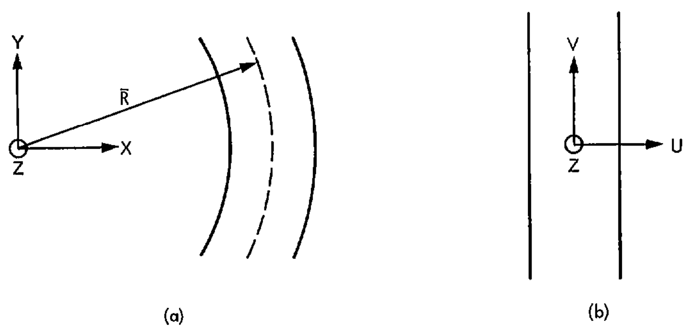

Fig. 2 Top view of a bent (2a) and a transformed straight (2b) fiber section

Appendix

Perturbation Analysis of a Bent Optical Fiber

The perturbation formula Eq. (12) is derived here. The equation that describes wave propagation in straight fiber is Eq. (4)

$$\frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2} + \left(\frac{\partial^2 \psi}{\partial z^2} + k^2 n_0^2 \psi\right) = 0 \qquad \text{(A-1)}$$

Propagation is in the v direction (Fig. 2b); hence the factor is $e^{i\beta v}$, where β is the propagation constant. With this, A-1 becomes

$$\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial z^2} + k^2 n_0^2\right)\psi = \beta^2 \psi \qquad \text{(A-2)}$$

or

$$H\psi = \beta^2 \psi \qquad \text{(A-3)}$$

where the operator H is as defined from (A-2) and (A-3).

For a bent fiber, the wave equation becomes (Eq. 11)

$$\frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2} + \left(\frac{\partial^2 \psi}{\partial v^2} + k^2 n^2(u)\psi\right)\left|\frac{dZ}{dw}\right|^2 = 0 \qquad \text{(A-4)}$$

Assuming a propagation constant is β', in this case we can write

$$\frac{\partial^2 \psi}{\partial u^2} + \left(\frac{\partial^2 \psi}{\partial z^2} + k^2 n^2(u)\psi\right)\left|\frac{dZ}{dw}\right| = \beta'^2 \psi \qquad \text{(A-5)}$$

or

$$H'\psi = \beta'\psi \qquad \text{(A-6)}$$

where

$$H' = \frac{\partial^2}{\partial u^2} + \left(\frac{\partial^2}{\partial z^2} + k^2 n^2(u)\right)\left|\frac{dZ}{dw}\right| \qquad \text{(A-7)}$$

If the bending radius is large, H' is not very different from H, and we can write

$$H' = H + H_1 \qquad \text{(A-8)}$$

where H_1 is "small" compared to H, and $\beta'^2 = \beta^2 + \delta(\beta^2)$, where $\delta(\beta^2)$ represents a small correction to the original β^2. The problem now is in the standard form of the perturbation theory (Ref. 12), and the correction to β^2 is given by

$$\delta(\beta^2) = \int_A H_1 |\psi|^2 \, dA \qquad \text{(A-9)}$$

where the area integral is evaluated over the cross section of the mode ψ. From Eqs. (A-8), (A-7) and (A-2) we have

$$\begin{aligned} H_1 &= H' - H \\ &= \frac{\partial^2}{\partial z^2}\left(\left|\frac{dZ}{dw}\right|^2 - 1\right) + k^2\left(n^2(u)\left|\frac{dZ}{dw}\right|^2 - n_0^2\right) \end{aligned} \qquad \text{(A-10)}$$

Now, since a mode can be interpreted as plane waves bouncing along the waveguide walls at grazing angles, the field variation in the transverse direction is much smaller than a wavelength, and thus

$$\frac{\partial^2 \psi}{\partial z^2} << k^2 \psi \qquad \text{(A-11)}$$

H_1 can thus be approximated to

$$H_1 = k^2\left(n^2(u)\left|\frac{dZ}{dw}\right|^2 - n_0^2\right) \qquad \text{(A-12)}$$

Substituting this into Eq. (A-9) gives Eq. (12).